

VECTOR BUNDLES OF RANK 2 COMPUTING CLIFFORD INDICES

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ABSTRACT. Clifford indices of vector bundles on algebraic curves were introduced in a previous paper of the authors. In this paper we study bundles of rank 2 which compute these Clifford indices. This is of particular interest in the light of recently discovered counterexamples to a conjecture of Mercat.

1. INTRODUCTION

Let C be a smooth irreducible curve of genus $g \geq 4$ defined over an algebraically closed field of characteristic zero. In describing the geometry of C , an important rôle is played by the classical *Clifford index* γ_1 , which may be defined as follows. For any line bundle L of degree d_L on C , consider $\gamma(L) := d_L - 2(h^0(L) - 1)$. Then we define

$$\gamma_1 := \min_L \{\gamma(L) \mid h^0(L) \geq 2, h^1(L) \geq 2\}$$

or equivalently

$$\gamma_1 := \min_L \{\gamma(L) \mid h^0(L) \geq 2, d_L \leq g - 1\}.$$

A line bundle L satisfying the conditions of the first definition is said to *contribute* to γ_1 ; if in addition $\gamma(L) = \gamma_1$, then L is said to *compute* γ_1 .

In a previous paper [8], we introduced Clifford indices for vector bundles, generalising the classical definition, as follows. For any vector bundle E of rank r_E and degree d_E on C , consider

$$\gamma(E) := \frac{1}{r_E}(d_E - 2(h^0(E) - r_E)) = \mu(E) - 2\frac{h^0(E)}{r_E} + 2.$$

We then define:

$$\gamma_n := \min_E \left\{ \gamma(E) \mid \begin{array}{l} E \text{ semistable of rank } n, \\ h^0(E) \geq n + 1, \mu(E) \leq g - 1 \end{array} \right\}$$

2000 *Mathematics Subject Classification*. Primary: 14H60; Secondary: 14F05, 32L10.

Key words and phrases. Semistable vector bundle, Clifford indices, gonality.

Both authors are members of the research group VBAC (Vector Bundles on Algebraic Curves). The second author would like to thank the Department Mathematik der Universität Erlangen-Nürnberg for its hospitality.

and

$$\gamma'_n := \min_E \left\{ \gamma(E) \mid \begin{array}{l} E \text{ semistable of rank } n, \\ h^0(E) \geq 2n, \mu(E) \leq g-1 \end{array} \right\}.$$

Note that γ_1 is just the classical Clifford index of the curve C and $\gamma'_1 = \gamma_1$. We say that E *contributes to* γ_n (respectively γ'_n) if E is semistable of rank n with $\mu(E) \leq g-1$ and $h^0(E) \geq n+1$ (respectively $h^0(E) \geq 2n$). If in addition $\gamma(E) = \gamma_n$ (respectively $\gamma(E) = \gamma'_n$), we say that E *computes* γ_n (respectively γ'_n).

Our object in the present paper is to study the bundles which compute γ_2 or γ'_2 . In order to describe our results, we need some further definitions. The *gonality sequence* $d_1, d_2, \dots, d_r, \dots$ of C is defined by

$$d_r := \min\{d_L \mid L \text{ a line bundle on } C \text{ with } h^0(L) \geq r+1\}.$$

We have always $d_r < d_{r+1}$ and $d_{r+s} \leq d_r + d_s$; in particular $d_n \leq nd_1$ for all n (see [8, Section 4]). We say that d_r *computes* γ_1 if $d_r \leq g-1$ and $d_r - 2r = \gamma_1$ and that C has *Clifford dimension* r if r is the smallest integer for which d_r computes γ_1 .

Following a section of preliminaries, we proceed in Section 3 to consider curves of Clifford dimension 2, in other words smooth plane curves. In this case, we can describe all the bundles computing either γ_2 or γ'_2 (Propositions 3.1, 3.2 and 3.3). In Section 4, we consider curves of Clifford dimension ≥ 3 ; these are also known as *exceptional curves* (see [2]). We determine all the bundles which compute γ_2 (Theorem 4.2 and Proposition 4.3). We are not able to determine all bundles computing γ'_2 except when $r = 3$ (Proposition 4.3) or $r = 4$ (Proposition 4.5), but we do describe how they arise when $r \geq 5$ and $g = 4r - 2$ (it is conjectured that all exceptional curves have genus $4r - 2$).

An interesting by-product of this investigation is that, for $r \geq 4$, we have $\gamma'_2 < \gamma_1$, yielding further counterexamples to Mercat's conjecture in rank 2 (see proposition 2.7) to add to those already described in [3] and [10]. In particular Proposition 4.5 and Theorem 4.8 give the following theorem.

Theorem 1.1. *Let C be a curve of Clifford dimension $r \geq 4$ of genus $g = 4r - 2$. Then there exists a stable bundle E of rank 2 and degree $\leq 4r - 3$ on C with $h^0(E) \geq 4$. In particular*

$$\gamma'_2 < \gamma_1.$$

In Section 5, we start the investigation of curves of Clifford dimension 1 by looking at hyperelliptic, trigonal and tetragonal curves. The most general result that we obtain is

Theorem 1.2. *Let C be a general tetragonal curve of genus $g \geq 8$. Then the tetragonal line bundle Q is unique and the only bundles computing γ_2 are*

- (1) $Q \oplus Q$;

(2) *possibly non-trivial extensions*

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q^* \rightarrow 0$$

$$\text{with } h^0(E) = h^0(Q) + h^0(K \otimes Q^*) = g - 1.$$

In particular $\gamma_2 = \gamma'_2 = \gamma_1$. When $g \geq 27$, bundles of type (2) do not exist.

(See Theorem 5.12 and Remarks 5.13 and 5.16.)

In Section 6 we study k -gonal curves for $k \geq 5$ and prove a similar result to Theorem 1.2 (Theorem 6.2 and Corollary 6.3).

Section 7 concerns general curves. For such curves it is conjectured (see [3]) that $\gamma'_2 = \gamma_1$ and this is certainly true for $g \leq 16$. We work out the possible bundles computing γ'_2 under this assumption (see Theorem 7.4).

In the final section we consider curves with $\gamma'_2 < \gamma_1$. Examples of such curves are known for all genera $g \geq 11$ (see [3], [10] and Theorem 1.1 above). In this case we show that all bundles computing γ'_2 are stable with $h^0(E) = 2 + s$, $s \geq 2$, and do not possess a line subbundle with $h^0 \geq 2$ (we refer to such bundles as bundles of type PR). We show that $s \leq \gamma'_2 - \frac{g}{2}$ (Proposition 8.2). In one case we get an almost complete description of the bundles computing γ'_2 .

Theorem 1.3. (Theorem 8.3) *Suppose $\gamma'_2 < \gamma_1$ and $d_4 = 2\gamma'_2 + 4$. Then the set of bundles of type PR with $s = 2$ which compute γ'_2 is in bijective correspondence with the set of line bundles*

$$U(d_4, 5) := \left\{ M \mid \begin{array}{l} d_M = d_4, \quad h^0(M) = 5, \\ S^2 H^0(M) \rightarrow H^0(M^2) \text{ not injective} \end{array} \right\}.$$

It is interesting to note that the condition that $S^2 H^0(M) \rightarrow H^0(M^2)$ be not injective can be restated in terms of Koszul cohomology and that there are close connections between the problems discussed here and the maximal rank conjecture (see [3]). Our results also have implications for the non-emptiness of higher rank Brill-Noether loci, but we have not developed this here because we have no “unexpected” results for general curves.

We suppose throughout that C is a smooth irreducible curve of genus $g \geq 4$ defined over an algebraically closed field of characteristic zero and that K denotes the canonical line bundle on C .

2. PRELIMINARIES

In this section, we recall a number of results from [8] and [9] and prove some additional lemmas. First recall from [8, Corollary 5.3, Lemma 2.2 and Theorem 5.2] that

$$(2.1) \quad \gamma_2 = \min \left\{ \gamma_1, \frac{d_2}{2} - 1 \right\} \quad \text{and} \quad \gamma_1 \geq \gamma'_2 \geq \min \left\{ \gamma_1, \frac{d_4}{2} - 2 \right\}.$$

Also [8, Lemma 4.6]

$$(2.2) \quad d_r \geq \min\{\gamma_1 + 2r, g + r - 1\}$$

Next we have from [9]

Lemma 2.1. *Any bundle computing γ_2 or γ'_2 is generated.*

We now recall that, for any generated line bundle L with $h^0(L) = 3$, one can define a vector bundle E_L of rank 2 by means of the evaluation sequence

$$(2.3) \quad 0 \rightarrow E_L^* \rightarrow H^0(L) \otimes \mathcal{O}_C \rightarrow L \rightarrow 0.$$

Lemma 2.2. *Let E_L be defined by (2.3). Then $h^0(E_L) \geq 3$. Moreover,*

- (i) *if $d_L \leq 2d_1$, then E_L is semistable;*
- (ii) *if $d_L < 2d_1$, then E_L is stable;*
- (iii) *if $d_L = d_2 < 2d_1$, then $h^0(E_L) = 3$.*

Proof. Dualising (2.3), we see that $h^0(E_L) \geq 3$ and E_L is generated. Since also $h^0(E_L^*) = 0$, it follows that any quotient line bundle of E_L has $h^0 \geq 2$ and therefore has degree $\geq d_1$. This gives (i) and (ii).

(iii) is a special case of [8, Theorem 4.15(a)]. \square

Lemma 2.3. *If E is a semistable bundle of rank 2 with $h^0(E) \geq 3$, then $d_E \geq d_2$. Moreover, if E computes γ_2 but not γ'_2 , then $E \simeq E_L$ for some line bundle L of degree $d_L = d_2$ with $h^0(L) = 3$.*

Proof. The first statement is the case $n = 2$ of [8, Proposition 4.11]. If E computes γ_2 but not γ'_2 , then certainly $h^0(E) = 3$; so $d_E \geq d_2$. Moreover E is generated by Lemma 2.1, so we have an exact sequence

$$0 \rightarrow L^* \rightarrow H^0(E) \otimes \mathcal{O} \rightarrow E \rightarrow 0,$$

where $L \simeq \det E$ is a generated line bundle of degree $d_L \geq d_2$ with $h^0(L) \geq 3$. In order to minimise $\gamma(E)$, we must take $d_L = d_2$ and then $h^0(L) = 3$ and $E \simeq E_L$. \square

Corollary 2.4. *If $d_2 \leq 2\gamma_1 + 2$, then the bundles computing γ_2 but not γ'_2 are precisely the bundles E_L for L a line bundle of degree d_2 with $h^0(L) = 3$.*

Proof. By the lemma, any bundle computing γ_2 but not γ'_2 has the form E_L . Since $d_2 \leq 2\gamma_1 + 2 \leq 2d_1 - 2$, it follows from Lemma 2.2 that E_L is stable and $h^0(E_L) = 3$. A direct computation using (2.1) gives $\gamma(E_L) = \gamma_2$. \square

In discussing γ'_2 , we shall make much use of the Lemma of Paranjape and Ramanan [14, Lemma 3.9] (see also [8, Lemma 4.8]), which we now state for the case of bundles of rank 2.

Lemma 2.5. *Let E be a vector bundle of rank 2 with $h^0(E) = 2 + s$ for some $s \geq 1$. Suppose that E has no line subbundle M with $h^0(M) \geq 2$. Then $h^0(\det E) \geq 2s + 1$ and, in particular, $d_E \geq d_{2s}$.*

As a complement to this lemma, we have

Lemma 2.6. *Suppose that E is a semistable bundle of rank 2 and degree $\leq 2g - 2$ which possesses a subbundle M with $h^0(M) \geq 2$. Then $\gamma(E) \geq \gamma_1$, with equality if and only if $\gamma(M) = \gamma(E/M) = \gamma_1$ and all sections of E/M lift to E .*

Proof. (This follows the proof of [8, Theorem 5.2].) By semistability, we have $d_M \leq g - 1$, so M contributes to γ_1 and $\gamma(M) \geq \gamma_1$. Moreover $d_{E/M} \geq d_M$, so, if $h^0(E/M) \leq h^0(M)$, then $\gamma(E/M) \geq \gamma(M) \geq \gamma_1$. Note also that

$$d_{K \otimes (E/M)^*} = 2g - 2 - d_E + d_M \geq d_M.$$

Hence, if $h^0(K \otimes (E/M)^*) \leq h^0(M)$, then $\gamma(E/M) = \gamma(K \otimes (E/M)^*) \geq \gamma_1$. If neither of these possibilities occurs, then E/M contributes to γ_1 , so again $\gamma(E/M) \geq \gamma_1$. The result now follows from the fact that $\gamma(E) \geq \frac{1}{2}(\gamma(M) + \gamma(E/M))$ with equality if and only if all sections of E/M lift to E . \square

In [12], V. Mercat made a conjecture concerning the number of sections that a semistable bundle E on a curve of given Clifford index may have. In the case of rank 2, this conjecture can be expressed as follows.

- (i) If $\gamma_1 + 2 \leq \mu(E) \leq g - 1$, then $\gamma(E) \geq \gamma_1$;
- (ii) if $\frac{\gamma_1 + 3}{2} \leq \mu(E) < \gamma_1 + 2$, then $h^0(E) \leq 3$;
- (iii) if $1 \leq \mu(E) < \frac{\gamma_1 + 3}{2}$, then $h^0(E) \leq 2$.

We observed in [8, Proposition 3.3] that this conjecture implies that $\gamma'_2 = \gamma_1$. In fact, we have

Proposition 2.7. *Mercat's conjecture for rank 2 holds if and only if $\gamma'_2 = \gamma_1$.*

Proof. By [8, Proposition 3.3], the conjecture implies that $\gamma'_2 = \gamma_1$. Conversely, suppose that $\gamma'_2 = \gamma_1$. Then certainly (i) holds. If $\mu(E) < \gamma_1 + 2$ and $h^0(E) \geq 4$, then E contributes to γ'_2 . On the other hand $\gamma(E) < \gamma_1 + 2 - 2 = \gamma_1$, a contradiction. Finally, suppose that $\mu(E) < \frac{\gamma_1 + 3}{2}$ and $h^0(E) \geq 3$. Then $\gamma(E) < \frac{\gamma_1 + 3}{2} - 1 = \frac{\gamma_1 + 1}{2}$; this contradicts (2.1). \square

Finally in this section we prove three lemmas which will be useful in determining when all sections of a quotient E/M lift to E .

Lemma 2.8. *There exists a non-trivial extension of vector bundles*

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$$

with the property that all sections of G lift to E if and only if the multiplication map

$$(2.4) \quad \alpha : H^0(G) \otimes H^0(K \otimes F^*) \rightarrow H^0(K \otimes F^* \otimes G)$$

is not surjective.

Proof. All sections of G lift to E if and only if the extension class is in the kernel of the canonical map

$$H^1(G^* \otimes F) \rightarrow \text{Hom}(H^0(G), H^1(F)).$$

We require the condition that this kernel is non-trivial which is the case if and only if the dual map (2.4) is not surjective. \square

Lemma 2.9. *Suppose G is a generated line bundle with $h^0(G) = 2$ and F is a stable vector bundle with $\mu(F) = \deg G$. Then we have*

$$h^0(F \otimes G) \geq \begin{cases} 2h^0(F) & \text{if } F \not\simeq G, \\ 3 & \text{if } F \simeq G. \end{cases}$$

Moreover,

$$\text{codim im}(\alpha) = \begin{cases} h^0(F \otimes G) - 2h^0(F) & \text{if } F \not\simeq G, \\ h^0(G \otimes G) - 3 & \text{if } F \simeq G. \end{cases}$$

Proof. Since G is generated with $h^0(G) = 2$, the first assertion follows from the base-point-free pencil trick. Note also that by the base-point-free pencil trick,

$$\ker(\alpha) \simeq H^0(G^* \otimes F^* \otimes K) \simeq H^1(F \otimes G)$$

and hence

$$\begin{aligned} \dim \text{im}(\alpha) &= 2[h^0(F) - d_F + r_F(g-1)] \\ &\quad - [h^0(F \otimes G) - d_F - r_F d_G + r_F(g-1)] \\ &= 2h^0(F) - h^0(F \otimes G) - d_F + r_F d_G + r_F(g-1). \end{aligned}$$

Since F is stable and $\mu(F) = \deg G$, we have $h^1(G \otimes F^* \otimes K) = 0$ if $F \not\simeq G$ and $= 1$ if $F \simeq G$. So

$$h^0(G \otimes F^* \otimes K) = \begin{cases} r_F(g-1) + r_F d_G - d_F & \text{if } F \not\simeq G, \\ r_F(g-1) + r_F d_G - d_F + 1 & \text{if } F \simeq G. \end{cases}$$

The result follows. \square

For the final lemma we need a definition which we shall use several times. A curve is said to be a *Petri curve* if the map

$$H^0(L) \otimes H^0(K \otimes L^*) \rightarrow H^0(K)$$

is injective for all line bundles L . It is important to note that the general curve of any genus is a Petri curve and that

$$(2.5) \quad d_r = g + r - \left\lfloor \frac{g}{r+1} \right\rfloor.$$

The equation (2.5) is straightforward from the definitions.

Lemma 2.10. *Let C be a Petri curve of genus g and Q a line bundle of degree d_1 computing γ_1 . Then*

- (1) $h^0(Q^2) = 3$ if g is even;
- (2) $h^0(Q^2) = 4$ if g is odd.

Proof. By Riemann-Roch,

$$\begin{aligned} h^0(K \otimes Q^{*2}) &= g - 2d_1 - 1 + h^0(Q^2) \\ &= \begin{cases} h^0(Q^2) - 3 & \text{if } g \text{ is even,} \\ h^0(Q^2) - 4 & \text{if } g \text{ is odd,} \end{cases} \end{aligned}$$

since $d_1 = \lceil \frac{g+3}{2} \rceil$. We claim now that $h^0(K \otimes Q^{*2}) = 0$ on a Petri curve giving the result.

To prove the claim, suppose that $0 \neq s \in H^0(K \otimes Q^{*2})$. Consider the commutative diagram

$$\begin{array}{ccc} H^0(Q) \otimes H^0(Q) & \longrightarrow & H^0(Q^2) \\ \text{id} \otimes (\cdot s) \downarrow & & \downarrow \cdot s \\ H^0(Q) \otimes H^0(K \otimes Q^*) & \longrightarrow & H^0(K). \end{array}$$

The left hand vertical homomorphism is clearly injective and for a Petri curve the bottom map as well. Since $h^0(Q) > 1$, the top horizontal map is not injective, a contradiction. \square

3. SMOOTH PLANE CURVES

Let C be a curve of Clifford dimension 2, in other words a smooth plane curve of degree $\delta \geq 5$. We recall (see, for example, [8, Section 8]) that

$$(3.1) \quad \gamma_1 = \delta - 4, \quad d_1 = \delta - 1, \quad d_2 = \delta, \quad d_3 = 2\delta - 2, \quad d_4 = 2\delta - 1.$$

Moreover, the hyperplane bundle H is the unique line bundle of degree δ with $h^0(H) = 3$ and H and $K \otimes H^* \simeq H^{\delta-4}$ are the only line bundles computing γ_1 . It is well known that C is projectively normal in \mathbb{P}^2 . So all multiplication maps $H^0(H^r) \otimes H^0(H^s) \rightarrow H^0(H^{r+s})$ are surjective.

Proposition 3.1. *For a smooth plane quintic the only bundle computing $\gamma_2 = \gamma'_2 = 1$ is $H \oplus H$.*

Proof. By (2.1) and (3.1), we see that $\gamma_2 = \gamma'_2 = 1$. Let E be a bundle computing γ_2 . By Lemma 2.3, we have $d_E \geq 5$. Moreover, $h^0(E) = \frac{d_E}{2} + 1$, implying that $h^0(E) \geq 4$ and E computes γ'_2 . If E has no line subbundle L with $h^0(L) \geq 2$, then, writing $h^0(E) = 2 + s$ with $s \geq 2$, it follows from Lemma 2.5 that $d_E \geq d_{2s} \geq d_4 + 2s - 4$. So $\gamma(E) \geq \frac{d_4}{2} - 2 = \delta - \frac{5}{2} > 1$, a contradiction.

So E must occur in an exact sequence

$$(3.2) \quad 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$$

with $h^0(M) \geq 2$. By Lemma 2.6, we must have $\gamma(M) = \gamma(N) = 1$ and hence $M \simeq N \simeq H$.

We need to show that the extension (3.2) must be trivial. By Lemma 2.8 it suffices to show that the map $H^0(H) \otimes H^0(K \otimes H^*) \rightarrow H^0(K)$ is surjective, which is the case, since $K \simeq H^2$. \square

Proposition 3.2. *For a smooth plane sextic the bundles computing $\gamma_2 = 2$ are E_H and $H \oplus H$. The only bundle computing $\gamma'_2 = 2$ is $H \oplus H$.*

Proof. The fact that $\gamma_2 = \gamma'_2 = 2$ follows from (2.1) and (3.1). By Corollary 2.4 and (3.1), E_H is the only bundle computing γ_2 but not γ'_2 .

Suppose now that E computes γ'_2 and write $h^0(E) = 2 + s$, $s \geq 2$. If E has no line subbundle with $h^0 \geq 2$, then Lemma 2.5 implies that $d_E \geq d_{2s}$. Moreover

$$d_{2s} \geq d_4 + 2s - 4 > 2s + 4$$

by (3.1). So $\gamma(E) > 2$, a contradiction. Thus E occurs in an exact sequence (3.2) with $h^0(M) \geq 2$; moreover, by Lemma 2.6, both M and N compute γ_1 .

Noting that H and H^2 are the only line bundles computing γ_1 , we have either $M \simeq N \simeq H$ or $M \simeq H$, $N \simeq H^2$. Since all sections of N must lift to E and $H^0(N) \otimes H^0(K \otimes M^*) \rightarrow H^0(K \otimes M^* \otimes N)$ is surjective in both cases, this allows only the split extension. Since E is semistable, only $E \simeq H \oplus H$ is possible. \square

Proposition 3.3. *Let C be a smooth plane curve of degree $\delta \geq 7$. Then $\gamma_2 < \gamma'_2 = \gamma_1$ and*

- (i) *E_H is the only bundle computing γ_2 ; moreover, E_H is stable with $h^0(E_H) = 3$;*
- (ii) *$H \oplus H$ is the only bundle computing γ'_2 .*

Proof. It follows from (2.1) and (3.1) that

$$\gamma_2 = \frac{\delta}{2} - 1 \quad \text{and} \quad \gamma'_2 = \gamma_1 = \delta - 4.$$

Hence $\gamma_2 < \gamma'_2$. (i) now follows from Corollary 2.4 and Lemma 2.2.

(ii) Suppose that E computes γ'_2 . If $h^0(E) = 2 + s$ with $s \geq 2$ and E has no line subbundle M with $h^0(M) \geq 2$, then by Lemma 2.5,

$$\gamma(E) \geq \frac{d_{2s}}{2} - s \geq \frac{d_4}{2} - 2 = \delta - \frac{5}{2} > \gamma_1,$$

a contradiction. It follows from Lemma 2.6 that E can be written as an extension (3.2) with $h^0(M) \geq 2$ and $\gamma(M) = \gamma(N) = \gamma_1$; moreover all sections of N lift to E . The proof is now completed exactly as for Proposition 3.2, noting that in this case either $N \simeq H$ or $N \simeq H^{\delta-4}$. \square

4. EXCEPTIONAL CURVES

In this section we consider curves of Clifford dimension ≥ 3 , in other words curves for which neither d_1 nor d_2 computes γ_1 .

Lemma 4.1. *Let C be a curve of Clifford dimension $r \geq 3$. Then*

$$d_1 > \frac{d_2}{2}, \quad \frac{d_2}{2} > \frac{d_3}{3} \quad \text{and} \quad \gamma_1 \geq \frac{d_2}{2} - 1.$$

The last inequality is strict for $r \geq 4$.

Proof. According to [2, Corollary 3.5], $d_r \geq 4r - 3$. Since d_1 does not compute γ_1 , we have

$$d_1 \geq \gamma_1 + 3 = d_r - 2r + 3.$$

Since $d_2 \leq d_r - r + 2$, this gives the first inequality. Since d_2 does not compute γ_1 , we have

$$d_2 \geq \gamma_1 + 5 = d_r - 2r + 5.$$

Using the fact that $d_3 \leq d_r - r + 3$, we obtain the second inequality. For the third inequality, we have

$$d_3 \leq d_r - r + 2 \leq 2d_r - 4r + 2 = 2\gamma_1 + 2,$$

since $d_r \geq 4r - 3 \geq 3r$. By the same reason this inequality is strict for $r \geq 4$. \square

Theorem 4.2. *Let C be of Clifford dimension $r \geq 3$. Then there is a semistable bundle E of rank 2 and degree d_2 with $h^0(E) = 3$. Any such bundle computes γ_2 , is of the form $E \simeq E_L$ and is stable.*

If $r \geq 4$ and $d_4 > d_2 + 2$, then these are the only bundles computing γ_2 .

Proof. Taking account of Lemma 4.1, the first part follows from Lemma 2.2 and (2.1).

If $r \geq 4$, we have $\gamma_1 > \frac{d_2}{2} - 1$ by Lemma 4.1. If $d_4 > d_2 + 2$, then $\frac{d_4}{2} - 2 > \frac{d_2}{2} - 1$. So $\gamma'_2 > \gamma_2$ by (2.1). Hence, by Corollary 2.4, these bundles of degree d_2 with $h^0 = 3$ are the only bundles computing γ_2 . \square

Let C be a curve of Clifford dimension 3. Then C is a complete intersection of 2 cubics in \mathbb{P}^3 [11]. The curve C is of genus 10 with

$$\gamma_1 = 3, \quad d_1 = 6, \quad d_2 = 8, \quad d_3 = 9 \quad \text{and} \quad d_4 = 12.$$

Let H denote the hyperplane bundle on C .

Proposition 4.3. *Let C be a curve of Clifford dimension 3. Then $\gamma_2 = \gamma'_2 = \gamma_1 = 3$ and the only bundles computing γ_2 are the bundles E_L of Theorem 4.2 and $H \oplus H$. The only bundle computing γ'_2 is $H \oplus H$.*

Proof. First, by (2.1) we have $\gamma_2 = \gamma'_2 = \gamma_1$. By Corollary 2.4, the bundles computing γ_2 but not γ'_2 are precisely the bundles E_L . Now suppose that E computes γ'_2 . If E has no line subbundle with $h^0 \geq 2$, then, by Lemma 2.5, $d_E \geq d_{2s}$ with $s \geq 2$ and hence

$$\gamma(E) \geq \frac{d_{2s}}{2} - s \geq \frac{d_4}{2} - 2 = 4,$$

a contradiction.

So, by Lemma 2.6, E fits into an extension (3.2) with $\gamma(M) = \gamma(N) = 3$. The only possibility is $M \simeq N \simeq H$ and all sections of N must lift to E . It follows from the projective normality of C in \mathbb{P}^3 and Lemma 2.8 that $E \simeq H \oplus H$. \square

Now let C be a curve of Clifford dimension $r = 4$. Then C has genus 14 and has a semicanonical projectively normal embedding into \mathbb{P}^4 given by the unique line bundle H of degree 13 computing γ_1 . In particular we have

$$\gamma_1 = 5, \quad d_1 = 8 \text{ and } d_4 = 13.$$

Lemma 4.4. $d_2 = 10, \quad d_3 = 12$ and $d_6 = 18$.

Proof. From Macdonald's secant plane formula (see [1, p. 351]) we see that there exist trisecants of C in \mathbb{P}^4 . Projecting from any such trisecant, we obtain a line bundle of degree 10 with $h^0 \geq 3$. Since no line bundle of degree < 13 can compute γ_1 , there does not exist a line bundle of degree 9 with $h^0 \geq 3$ or of degree 11 with $h^0 \geq 4$. Hence $d_2 = 10$ and $d_3 = 12$. The last assertion follows by duality. \square

Proposition 4.5. *Let C be a curve of Clifford dimension 4. Then*

- (i) $\gamma_2 = 4$ and is computed by a bundle of the form E_L with L a line bundle of degree 10 with $h^0(L) = 3$ and by no other bundles;
- (ii) $\gamma'_2 = \frac{9}{2}$ and is computed by a unique bundle E of degree 13 with $\det E = H$.

Proof. (i) Since $d_4 > d_2 + 2$, this is included in Theorem 4.2.

(ii) According to [2] C is contained in a K3-surface X which is embedded by a complete linear system into \mathbb{P}^4 . Hence X is the complete intersection of a quadric and a cubic in \mathbb{P}^4 . It follows from [5, Remark 3.4] and the uniqueness of H that there exists a unique stable rank-2 bundle E of degree 13 with $h^0(E) = 4$ and that $\det E = H$. Certainly $\gamma(E) = \frac{9}{2}$. So by (2.1), $\gamma'_2 = \frac{9}{2}$ and is computed by E .

Let E be any bundle of degree > 13 computing γ'_2 . Write $h^0(E) = 2 + s$ with $s \geq 3$ and $d_E = 9 + 2s$. If E has a subbundle M with $h^0(M) \geq 2$, then, by Lemma 2.6, $\gamma(E) \geq \gamma_1$, a contradiction. So Lemma 2.5 applies to give

$$d_E \geq d_{2s}.$$

This fails for $s = 3$, since $d_6 = 18$, and hence for all $s \geq 3$. \square

Remark 4.6. Since $\gamma'_2 < \gamma_1$, this gives a new example of a curve for which Mercat's conjecture fails.

Now suppose that C is of Clifford dimension $r \geq 5$ and of genus $g = 4r - 2$. This implies that

$$(4.1) \quad d_r = 4r - 3, \quad \gamma_1 = 2r - 3, \quad d_1 = 2r \text{ and } d_{r-1} = 4r - 4.$$

For the first equation note that $d_r \leq g - 1$ by the definition of γ_1 and $d_r \geq 4r - 3$ by [2, Corollary 3.5]. The last 2 equalities come from the fact that the Clifford dimension of C is r . According to [2], there exist curves of this type for any r . Let H denote the line bundle computing γ_1 . Then H gives a non-degenerate embedding of C into \mathbb{P}^r (note that C in \mathbb{P}^r must be smooth, since otherwise projection from a singular point would give Clifford dimension $< r$).

Now consider the canonical map

$$S^2 H^0(H) \rightarrow H^0(H^2).$$

We have $\dim S^2 H^0(H) = \frac{1}{2}(r+1)(r+2)$ and $h^0(H^2) \leq 4r - 2$. It follows that C is contained in at least

$$\frac{1}{2}(r+1)(r+2) - 4r + 2 = \frac{1}{2}(r^2 - 5r + 6) = \binom{r-2}{2}$$

independent quadrics.

Therefore a dimensional computation shows that C is contained in a quadric of rank ≤ 5 . In fact, C cannot be contained in a quadric of rank ≤ 4 , since otherwise the systems of maximal linear subspaces on the quadric would give pencils L_1, L_2 on C such that $H \simeq L_1 \otimes L_2$. This is impossible since $d_r < 2d_1$. So C lies on a quadric q of rank 5. As such, q contains a 3-dimensional system of $(r-3)$ -planes all of which contain the vertex of q .

If C does not meet the vertex of q , then through each point of C there is only a 1-dimensional system of $(r-3)$ -planes. So there exists an $(r-3)$ -plane not meeting C .

If C does meet the vertex, it does so in at most a finite number of points. It follows that by projection we can obtain a non-degenerate morphism $C \rightarrow \mathbb{P}^s$ with $s < r$ such that the image of C in \mathbb{P}^s is contained in a quadric q' of rank 5 and does not meet the vertex of q' . So there exists an $(s-3)$ -plane on q' not meeting C .

Lemma 4.7. *There exists a non-degenerate morphism $C \rightarrow \mathbb{P}^s$ for some $s \leq r$ such that, if H' denotes the hyperplane bundle of C in \mathbb{P}^s , then there exists a 3-dimensional subspace W of $H^0(H')$ such that*

- W generates H' ;
- the kernel N of the linear map $W \otimes H^0(H') \rightarrow H^0(H'^2)$ has dimension ≥ 4 .

Proof. From the previous discussion we obtain the morphism $C \rightarrow \mathbb{P}^s$ such that C in \mathbb{P}^s is contained in a quadric q' of rank 5 and there exists an $(s-3)$ -plane Π on q' not meeting C .

Now let W be the 3-dimensional subspace of $H^0(H')$ which annihilates the $(s-2)$ -dimensional subspace of $H^0(H')^*$ defined by Π . Since Π does not meet C , it follows that W generates H' . Moreover, since Π lies on q' , the image of $W \otimes H^0(H')$ in $S^2 H^0(H')$ contains the 1-dimensional subspace corresponding to the quadric q' . It follows that

the image of N in $S^2H^0(H')$ has dimension at least 1. Since the kernel of the map $N \rightarrow S^2H^0(H')$ is $\bigwedge^2 W$, it follows that $\dim N \geq 4$. \square

Theorem 4.8. *Let C be a curve of Clifford dimension $r \geq 5$ of genus $g = 4r - 2$. Then there exists a stable bundle E of rank 2 and degree $\leq 4r - 3$ on C with $h^0(E) \geq 4$. In particular*

$$\gamma'_2 \leq \gamma(E) < \gamma_1.$$

Proof. Let H' and W be as in Lemma 4.7. Define the bundle E^* by the exact sequence

$$(4.2) \quad 0 \rightarrow E^* \rightarrow W \otimes \mathcal{O}_C \rightarrow H' \rightarrow 0.$$

Tensoring by H' and noting that $H' \simeq \det E$ and hence $E^* \otimes H' \simeq E$, we obtain

$$0 \rightarrow E \rightarrow W \otimes H' \rightarrow H'^2 \rightarrow 0.$$

This implies $H^0(E) \simeq N$ and so $h^0(E) \geq 4$. Note that $d_E = d_{H'} \leq 4r - 3$.

From (4.2) we get that $h^0(E^*) = 0$ and E is generated. Hence any quotient line bundle L of E has $h^0(L) \geq 2$. So $d_L \geq d_1 = 2r$ and E is stable. Note that

$$\gamma(E) \leq \frac{4r-3}{2} - 2 = 2r - \frac{7}{2} < \gamma_1.$$

\square

Corollary 4.9. *Let C be a curve of Clifford dimension $r = 5$. Then*

$$6 \leq \gamma'_2 \leq \frac{13}{2}.$$

Moreover, if a bundle E computes γ'_2 , then $h^0(E) = 4$.

Proof. According to [2], C is of genus 18. The bundle E constructed in the theorem has $\gamma(E) \leq \frac{13}{2}$. According to (4.1), $d_4 = 16$. So by (2.1), $\gamma'_2 \geq 6$.

Let F be a bundle computing γ'_2 . If $h^0(F) = 2 + s$ with $s \geq 3$, then Lemmas 2.5 and 2.6 imply that $d_F \geq d_{2s}$. Since $\gamma'_2 \leq \frac{13}{2}$, we have $d_F \leq 13 + 2s$. This contradicts the fact that $d_6 \geq 20$, which follows by duality from the fact that $d_4 = 16$. \square

Remark 4.10. A curve C of Clifford dimension 5 has $\gamma'_2 = 6$ if and only if there exists a line bundle M of degree $d_4 = 16$ with $h^0(M) = 5$ and $S^2H^0(M) \rightarrow H^0(M^2)$ non-injective. This follows from [5, Theorem 3.2] and Proposition 8.6 below. From our previous discussion, the curve C in \mathbb{P}^5 lies on a number of quadric cones; if it passes through the vertex of one of these cones, then projection from this vertex gives the required bundle M . We do not know whether this can happen.

5. HYPERELLIPTIC, TRIGONAL AND TETRAGONAL CURVES

From now on, we consider curves of Clifford dimension 1. These are also known as k -gonal curves, where $k = d_1 = \gamma_1 + 2$. In this section, we study the cases $2 \leq k \leq 4$, in other words hyperelliptic, trigonal and tetragonal curves. For these curves, it follows from (2.1) that

$$\gamma_2 = \gamma'_2 = \gamma_1.$$

For hyperelliptic curves it is well known that the only bundles computing $\gamma_2 = \gamma'_2 = 0$ are the bundles

$$H^r \oplus H^r$$

where H is the hyperelliptic line bundle and $1 \leq r \leq \frac{g-1}{2}$ [15, Proposition 2].

Proposition 5.1. *Let C be a trigonal curve of genus $g \geq 5$ and denote by T the trigonal line bundle. Then the bundles computing γ_2 are*

- $T \oplus T$;
- possibly stable bundles fitting into a non-trivial extension

$$(5.1) \quad 0 \rightarrow T \rightarrow E \rightarrow K \otimes T^* \rightarrow 0$$

$$\text{with } h^0(E) = h^0(T) + h^0(K \otimes T^*) = g.$$

In particular every bundle computing γ_2 also computes γ'_2 .

Proof. If E computes γ_2 , then $d_E \geq d_2$ by Lemma 2.3. Hence, since $\gamma_2 = 1$,

$$h^0(E) = \frac{d_E}{2} + 1 \quad \text{and} \quad 6 \leq d_E \leq 2g - 2$$

with d_E even.

If E has no line subbundle M with $h^0(M) \geq 2$, then, by Lemma 2.5, $d_E \geq d_r$ with $r = 2(\frac{d_E}{2} - 1) = d_E - 2$. A simple numerical calculation using (2.2) gives a contradiction.

So we have an exact sequence

$$(5.2) \quad 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$$

with $h^0(M) \geq 2$. Since $d_M \leq g - 1$ by semistability, we have (see [8, Remark 4.5(b)])

$$h^0(M) \leq \frac{d_M}{3} + 1.$$

If $d_N \leq g - 1$, then $h^0(N) \leq \frac{d_N}{3} + 1$. So $h^0(E) \leq h^0(M) + h^0(N) \leq \frac{d_E}{3} + 2$. This contradicts $h^0(E) = \frac{d_E}{2} + 1$ except when $d_E = 6$ and in this case $M \simeq N \simeq T$. If $d_E = 6$ and (5.2) does not split, then according to Lemma 2.9 we have $h^0(T^2) \geq 4$, which is impossible since $d_{T^2} = 6$ and $d_3 \geq 7$.

If $d_N > g - 1$, then

$$\begin{aligned} h^0(N) &= h^1(N) + d_N - g + 1 \\ &\leq \frac{d_{K \otimes N^*}}{3} + d_N - g + 2 = \frac{2d_N - g + 4}{3}. \end{aligned}$$

So $h^0(E) \leq \frac{d_M + 2d_N - g + 4}{3} + 1$, i.e.

$$\frac{d_M + d_N}{2} = \frac{d_E}{2} \leq \frac{d_M + 2d_N - g + 4}{3}$$

which is equivalent to

$$d_N - d_M \geq 2g - 8.$$

This requires $d_M = 3$, $d_N = 2g - 5$ with $h^0(M) = h^0(K \otimes N^*) = 2$. Hence $M \simeq K \otimes N^* \simeq T$ and (5.2) becomes (5.1).

If E is not stable, then E occurs in an extension

$$0 \rightarrow M' \rightarrow E \rightarrow N' \rightarrow 0$$

with $\frac{d_E}{2} \leq d_{M'} < 2g - 5$. The same calculation to estimate $h^0(E)$ gives $h^0(E) < \frac{d_E}{2} + 1$, a contradiction. \square

Remark 5.2. Extensions (5.1) exist if and only if the multiplication map

$$(5.3) \quad H^0(K \otimes T^*) \otimes H^0(K \otimes T^*) \rightarrow H^0(K^2 \otimes T^{*2})$$

is not surjective. For $g \geq 17$ these extensions do not exist. (The condition $g \geq 17$ is probably not best possible.)

Proof. Condition (5.3) follows from Lemma 2.8. The last assertion follows from [4, Theorem 2(a)]. For this we have to show that $K \otimes T^*$ is very ample. In fact for any $p, q \in C$,

$$h^0(K \otimes T^*(-p - q)) = h^1(T(p + q)) = h^1(T) - 2 = h^0(K \otimes T^*) - 2,$$

since $h^0(T(p + q)) = h^0(T) = 2$. \square

Remark 5.3. Let M and N be the trigonal line bundles on a general curve of genus 4. Then we have $M \otimes N \simeq K$. A modified version of the previous argument shows that the only semistable bundles computing γ_2 are

$$M \oplus M, \quad N \oplus N \quad \text{and} \quad M \oplus N.$$

For the trigonal curves of genus 4 for which there exists only one trigonal line bundle M we have also a unique semistable bundle E with $h^0(E) = 4$ which occurs as a non-trivial extension $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$.

Lemma 5.4. *Let C be a tetragonal curve. Then $d_2 \geq 6$ and there exists a bundle E computing γ_2 with $h^0(E) = 3$ if and only if $d_2 = 6$.*

Proof. We cannot have $d_2 \leq 5$, since otherwise $\gamma_1 \leq 5 - 4 = 1$. A semistable bundle E with $h^0(E) \geq 3$ has $d_E \geq d_2$ by Lemma 2.3. If $h^0(E) = 3$, then $\gamma(E) \geq \frac{1}{2}(d_2 - 2)$. So E can compute $\gamma_2 = 2$ only if $d_E = d_2 = 6$. For the existence of E we take $E = E_L$ where L is a line bundle of degree d_2 with $h^0(L) = 3$. \square

Remark 5.5. The condition $d_2 = 6$ is satisfied when $g = 5$ or 6 and for all bielliptic curves.

Lemma 5.6. *Let C be a tetragonal curve. If there exists a bundle E computing γ'_2 with no line subbundle M with $h^0(M) \geq 2$, then $g = 5$ and E fits into an exact sequence*

$$(5.4) \quad 0 \rightarrow M \rightarrow E \rightarrow K \otimes M^* \rightarrow 0$$

with $d_M = 2$ and $h^0(M) = 1$. Moreover, when $g = 5$, there exist non-trivial extensions (5.4) for which all sections of $K \otimes M^$ lift and all such E are semistable.*

Proof. Write $h^0(E) = 2 + s$ with $s \geq 2$. By Lemma 2.5, $d_E \geq d_{2s} \geq d_4 + 2s - 4$. So

$$\gamma(E) \geq \frac{1}{2}(d_4 + 2s - 4 - 2s) = \frac{d_4}{2} - 2.$$

This implies $d_4 \leq 8$. A line bundle L of degree d_4 with $h^0(L) = 5$ has $h^1(L) \geq 2$ if $d_4 \leq g + 2$ and therefore contributes to the Clifford index. This gives a contradiction for $g \geq 6$.

For $g = 5$, we must have $d_4 = 8$, $d_E = 8$ and $s = 2$. Moreover, by Lemma 2.5, we have $h^0(\det E) \geq 5$. So $\det E \simeq K$. By [13] E has a line subbundle M of degree ≥ 2 with $h^0(M) \leq 1$. So $h^0(E/M) \geq 3$, which implies $d_{E/M} \geq d_2 = 6$. Since $E/M \simeq K \otimes M^*$, we obtain (5.4) as required.

According to Lemma 2.8 there exist non-trivial extensions (5.4) for which all sections of $K \otimes M^*$ lift if and only if

$$H^0(K \otimes M^*) \otimes H^0(K \otimes M^*) \rightarrow H^0(K^2 \otimes M^{*2})$$

is not surjective. The map factors through $S^2 H^0(K \otimes M^*)$ which has dimension 6. On the other hand, $h^0(K^2 \otimes M^{*2}) = 8$.

If E is not semistable, then E possesses a line subbundle M' of degree 5. Now $h^0(M') \leq 2$ and $h^0(E/M') \leq 1$. So $h^0(E) \leq 3$, a contradiction. \square

We now consider tetragonal curves of genus 5 starting with curves which satisfy the Petri condition.

Proposition 5.7. *Let C be a Petri curve of genus 5. Then the bundles computing γ_2 are*

- (1) E_L with $d_L = d_2 = 6$;
- (2) $Q \oplus Q'$, where Q and Q' are tetragonal line bundles;

- (3) *for each tetragonal line bundle Q a unique bundle E which is a non-trivial extension*

$$(5.5) \quad 0 \rightarrow Q \rightarrow E \rightarrow Q \rightarrow 0$$

with $h^0(E) = 4$:

- (4) *for each tetragonal line bundle Q a unique bundle E which is a non-trivial extension*

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q^* \rightarrow 0$$

with $h^0(E) = 4$:

- (5) *possibly stable bundles fitting into a non-trivial extension (5.4).*

All these bundles compute γ'_2 except for those of type (1).

Proof. The bundles E_L and $Q \oplus Q'$ have $\gamma(E) = \gamma_2$ by direct numerical calculations.

If E computes γ_2 and possesses a line subbundle M with $h^0(M) \geq 2$, then by Lemma 2.6 there is a non-trivial extension $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ with tetragonal line bundles $M = Q$ and $N = Q'$. By Lemma 2.8, there exist such extensions for which all sections of Q' lift if and only if the map

$$(5.6) \quad H^0(Q') \otimes H^0(K \otimes Q^*) \rightarrow H^0(K \otimes Q^* \otimes Q')$$

is not surjective. By Lemmas 2.9 and 2.10 this happens if and only if either $Q' \simeq Q$ or $Q' \simeq K \otimes Q^*$.

By Lemma 2.10, the Petri condition implies that $Q \not\simeq K \otimes Q^*$. This gives the cases (3) and (4). The uniqueness statement follows from Lemma 2.9.

If E does not possess a line subbundle M with $h^0(M) \geq 2$, we obtain an extension (5.4) by Lemma 5.6. Any strictly semistable bundles which are included among these are already contained in (3) or (4). \square

Remark 5.8. It is not clear whether the bundles in (5) actually exist.

Remark 5.9. If C is not Petri and $Q^2 \simeq K$, then there exists a family of bundles E parametrized by \mathbb{P}^1 which occur as non-trivial extensions (5.5). Otherwise the results are the same as in Proposition 5.7.

Proposition 5.10. *For a tetragonal curve of genus $g = 6$ the bundles computing γ_2 are*

- (1) E_L with $d_L = d_2 = 6$;
- (2) $Q \oplus Q'$, where Q and Q' are tetragonal line bundles;
- (3) if C is not a Petri curve, bundles E of degree 8 with $h^0(E) = 4$ given by non-trivial extensions

$$0 \rightarrow Q \rightarrow E \rightarrow Q \rightarrow 0,$$

where Q is a tetragonal line bundle and $h^0(Q^2) = 4$;

- (4) *stable bundles E of degree 10 with $h^0(E) = 5$ given by non-trivial extensions*

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q'^* \rightarrow 0$$

where Q and Q' are tetragonal line bundles (such bundles exist when $Q \simeq Q'$).

All these bundles compute γ'_2 except for those of type (1).

Proof. The bundles E_L and $Q \oplus Q'$ certainly have $\gamma(E) = \gamma_2 = 2$.

In view of Lemmas 5.6 and 2.6, all other bundles E computing γ_2 arise as extensions

$$0 \rightarrow Q \rightarrow E \rightarrow N \rightarrow 0$$

where Q is a tetragonal line bundle and N is either a tetragonal line bundle Q' or the Serre dual $K \otimes Q'^*$ of a tetragonal line bundle Q' . In both cases all sections of N must lift to E .

It is easy to check that $d_4 = 9$. Hence, when $N \simeq Q'$, it follows from Lemma 2.9 that, if the extension is non-trivial, then E can exist only if $Q' \simeq Q$ and $h^0(Q^2) = 4$. This cannot happen on a Petri curve by Lemma 2.10. This gives (3).

The only remaining case is (4). Here the extension must be non-trivial, since E is semistable. Conversely, all non-trivial extensions of this type yield semistable bundles E and all those with $h^0(E) = 5$ are in fact stable. Existence when $Q \simeq Q'$ follows from Lemma 2.8 and the fact that $S^2 H^0(K \otimes Q^*)$ has dimension 6, while $h^0(K^2 \otimes Q^{*2}) = 7$. \square

Proposition 5.11. *Let C be a tetragonal curve of genus $g = 7$ such that $d_2 = 7$. Then the bundles computing γ_2 are*

- (1) $Q \oplus Q'$, where Q and Q' are tetragonal line bundles;
- (2) *possibly non-trivial extensions*

$$0 \rightarrow Q \rightarrow E \rightarrow Q \rightarrow 0,$$

where Q is a tetragonal line bundle with $h^0(E) = 4$ (such extensions exist if and only if $h^0(Q^2) = 4$);

- (3) *possibly non-trivial extensions*

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q'^* \rightarrow 0$$

with $h^0(E) = h^0(Q) + h^0(K \otimes Q'^) = 6$.*

In particular, every bundle computing γ_2 also computes γ'_2 .

Proof. By Lemma 5.4 every bundle E computing γ_2 has $h^0(E) \geq 4$. Hence by Lemma 5.6, E possesses a line subbundle M with $h^0(M) \geq 2$. Writing $N = E/M$, it follows by Lemma 2.6 that $\gamma(M) = \gamma(N) = \gamma_1 = 2$ and all sections of N lift to E . Since $d_M \leq \mu(E) \leq g - 1$, M must be a tetragonal line bundle Q . If $d_N \leq g - 1$, the same holds for N ; if $d_N > g - 1$, then N must be the Serre dual $K \otimes Q'^*$ of a tetragonal line bundle Q' .

Since $d_3 = 8$ by Serre duality, we have $h^0(Q \otimes Q') \leq 4$. Lemma 2.9 now implies (2). \square

Theorem 5.12. *Let C be a tetragonal curve of genus $g \geq 8$ such that the tetragonal line bundles are the only line bundles of degree $\leq g - 1$ which compute γ_1 . Then the tetragonal line bundle Q is unique and the only bundles computing γ_2 are*

- (1) $Q \oplus Q$;
- (2) possibly non-trivial extensions

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q^* \rightarrow 0$$

$$\text{with } h^0(E) = h^0(Q) + h^0(K \otimes Q^*) = g - 1.$$

In particular, every bundle computing γ_2 also computes γ'_2 .

Proof. If $g \geq 9$, then $d_3 \geq 9$ by hypothesis. If $g = 8$, the same holds by Serre duality. Hence, if Q and Q' are tetragonal line bundles, we have $h^0(Q \otimes Q') = 3$. So by the base-point-free pencil trick, $Q \simeq Q'$. Moreover, by Lemma 2.9 there does not exist a non-trivial extension $0 \rightarrow Q \rightarrow E \rightarrow Q \rightarrow 0$ for which all sections of Q lift to E .

Since $d_2 \geq 7$, the bundles E_L with $d_L = d_2$ and $h^0(L) = 3$ do not compute γ_2 . The proof now proceeds in the same way as the proof of the previous proposition. \square

Remark 5.13. The hypothesis is satisfied for a general tetragonal curve (see [8, Remark 4.5(c)]).

Example 5.14. The normalisation of a plane curve of degree 6 with 2 nodes is a tetragonal curve of genus 8 with $d_2 = 6$ and $d_3 = 8$. It possesses 2 tetragonal line bundles Q and Q' ; the hyperplane bundle H also computes γ_1 . The 2 tetragonal line bundles Q and Q' are non-isomorphic and $h^0(Q^2) = h^0(Q'^2) = 3$. The bundle E_H computes γ_2 but not γ'_2 . Moreover, the only bundles computing γ'_2 are

$$Q \oplus Q, \quad Q \oplus Q', \quad Q' \oplus Q', \quad H \oplus H$$

and unique non-trivial extensions

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Q' \rightarrow E \rightarrow K \otimes Q' \rightarrow 0.$$

To see this, we have to show according to Lemma 2.8 that the map $H^0(K \otimes Q^*) \otimes H^0(K \otimes Q^*) \rightarrow H^0(K^2 \otimes Q^* \otimes Q^*)$ is surjective and that the map $H^0(K \otimes Q^*) \otimes H^0(K \otimes Q^*) \rightarrow H^0(K^2 \otimes Q^{*2})$ and the analogous map for Q replaced by Q' have 1-dimensional cokernel.

The linear series $|K \otimes Q^*|$ is cut by conics through the node corresponding to Q' with a similar statement for $|K \otimes Q'^*|$. Moreover, $|K^2 \otimes Q^* \otimes Q'^*|$ is cut by quartics through both nodes. It is easy to see that this gives the surjectivity. On the other hand, the linear series $|K^2 \otimes Q^{*2}|$ has dimension 12, while the linear system of quartics with a double point at the corresponding node is only 11-dimensional.

In the same way one checks that there are no non-trivial extensions $0 \rightarrow H \rightarrow E \rightarrow H \rightarrow 0$ or $0 \rightarrow H \rightarrow E \rightarrow K \otimes H^* \rightarrow 0$ such that E computes γ'_2 .

Example 5.15. A smooth $(4, 4)$ -curve on a smooth quadric surface is a tetragonal curve of genus 9 with $d_2 = 7$, $d_3 = 8$ and possessing 2 tetragonal line bundles Q and Q' . These bundles are non-isomorphic and $h^0(Q^2) = h^0(Q'^2) = 3$. The hyperplane bundle H also computes γ_1 . The bundles computing γ'_2 are exactly as in the previous example.

This follows from the fact that $|K \otimes Q^*|$ and $|K \otimes Q'^*|$ are cut respectively by $(1, 2)$ - and $(2, 1)$ -curves on the quadric while $|K^2 \otimes Q^* \otimes Q'^*|$ is cut by $(3, 3)$ -curves. On the other hand, the codimension of the linear system of $(4, 2)$ -curves in $|K^2 \otimes Q^{*2}|$ is again 1. Similarly $|H|$ is cut by $(1, 1)$ -curves and $|H^2|$ by $(2, 2)$ -curves.

Remark 5.16. Under the hypotheses of Theorem 5.12, extensions of type (2) exist if and only if the multiplication map

$$H^0(K \otimes Q^*) \otimes H^0(K \otimes Q^*) \rightarrow H^0(K^2 \otimes Q^{*2})$$

is not surjective. For a general tetragonal curve of genus $g \geq 27$ there are no such bundles. (The condition $g \geq 27$ is probably not best possible.)

Proof. The statement concerning existence is Lemma 2.8.

The surjectivity of the multiplication map follows from [4, Theorem 2(a)]. For this we have to show that $K \otimes Q^*$ is very ample. The proof is the same as in the proof of Remark 5.2. \square

Finally in this section, we look at tetragonal curves which are in some sense at the opposite extreme from the general ones, namely bielliptic curves. For this case we list some bundles computing γ_2 and γ'_2 , but we do not know whether the list is complete.

Proposition 5.17. *Let $\pi : C \rightarrow C'$ be a double covering of an elliptic curve for which C has genus g and $\gamma_1 = 2$. Then, if E' is a semistable rank-2 bundle on C' of degree d' , $3 \leq d' \leq g - 1$, the pull-back π^*E' computes γ_2 . For $d' \geq 4$ it also computes γ'_2 .*

Proof. The pull-back π^*E' is semistable of degree $2d'$ and $h^0(\pi^*E') \geq d'$. So

$$\gamma(\pi^*E') \leq \frac{1}{2}(2d' - 2(d' - 2)) = 2.$$

Since $\gamma_2 = 2$, we must have equality. For γ'_2 we need also $d' \geq 4$. \square

6. k -GONAL CURVES FOR $k \geq 5$

Proposition 6.1. *Suppose $d_2 = 2d_1$. Then $\gamma_2 = \gamma'_2 = \gamma_1$ and all bundles computing γ_2 compute γ'_2 . Moreover, there exists a unique semistable bundle E of degree d_2 computing γ'_2 and*

$$E \simeq Q \oplus Q,$$

where Q is the unique line bundle of degree d_1 computing γ_1 .

Proof. If $d_2 = 2d_1$, then $\gamma_1 \leq d_1 - 2 < \frac{d_2}{2} - 1$. So $\gamma_2 = \gamma_1$ and hence also $\gamma'_2 = \gamma_1$.

If $h^0(E) = 3$, then $d_E \geq d_2$ by Lemma 2.3. This gives $\gamma(E) > \gamma_1$, a contradiction.

Let E be a semistable bundle of rank 2 and degree d_2 with $h^0(E) \geq 4$. Since $d_4 > d_2$, Lemma 2.5 implies that E has a line subbundle M with $h^0(M) \geq 2$ and hence $d_M \geq d_1$. Lemma 2.6 now gives an extension

$$(6.1) \quad 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$$

with M and N line bundles of degree d_1 computing γ_1 and all sections of N lift to E . Since $d_{M \otimes N} = d_2$, we have $h^0(M \otimes N) \leq 3$ which implies that $M \simeq N$ and the extension (6.1) splits by Lemma 2.9. So $M \simeq Q$ is unique (provided it exists) and $E \simeq Q \oplus Q$. Clearly $\gamma(Q \oplus Q) = \gamma'_2$.

For existence of Q , we need to know that d_1 computes γ_1 . This holds because $d_2 = 2d_1$ cannot hold on a curve of Clifford dimension ≥ 2 . This is obvious for smooth plane curves; for exceptional curves, we have (see the proof of Lemma 4.1)

$$(6.2) \quad 2d_1 - d_2 \geq 2d_r - 4r + 6 - (d_r - r + 2) = d_r - 3r + 4 \geq r + 1.$$

□

Theorem 6.2. *Suppose $d_2 = 2d_1$ and the line bundle Q of degree d_1 is the only line bundle of degree $\leq g - 1$ computing γ_1 . Then the bundles computing $\gamma_2 = \gamma'_2$ are*

- (1) $Q \oplus Q$;
- (2) possibly non-trivial extensions

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q^* \rightarrow 0$$

$$\text{with } h^0(E) = h^0(Q) + h^0(K \otimes Q^*) = g + 3 - d_1.$$

Proof. Let E be a bundle computing $\gamma'_2 = \gamma_2$ and write $h^0(E) = 2 + s$, $s \geq 2$. If E has no line subbundle with $h^0 \geq 2$, then by Lemma 2.5, $d_E \geq d_{2s}$. So

$$\gamma(E) \geq \frac{d_{2s}}{2} - s \geq \frac{d_2}{2} - 1 = d_1 - 1 > \gamma_1,$$

a contradiction. So by Lemma 2.6 there exists an extension

$$0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$$

with M and N line bundles computing γ_1 and such that any section of N lifts to E . The only possibilities are $M \simeq N \simeq Q$ and $M \simeq Q$, $N \simeq K \otimes Q^*$. The rest is contained in Proposition 6.1. □

Corollary 6.3. *For $k \geq 5$, let C be a general k -gonal curve of genus $g > \max\{3k^2 - 8k + 7, 46\}$. Then the only bundle computing $\gamma_2 = \gamma'_2$ is $Q \oplus Q$, where Q is the unique line bundle of degree $\leq g - 1$ computing γ_1 .*

Proof. We use the fact (see [6, Theorem 3.1]) that

$$d_r = kr \quad \text{for} \quad 1 \leq r \leq \frac{1}{k-2} \left\lfloor \frac{g-4}{2} \right\rfloor.$$

In particular we have $d_2 = 2d_1$ for $g \geq 4k-4$ which holds by hypothesis. Now write

$$r_0 := \left\lfloor \frac{1}{k-2} \left\lfloor \frac{g-4}{2} \right\rfloor \right\rfloor.$$

A line bundle L with $d_L > kr_0$ has $h^0(L) - 1 \leq d_L - kr_0 + r_0$. Given that $d_L \leq g-1$, this implies that

$$\gamma(L) \geq g-1-2(g-1-kr_0+r_0) = 2(k-1)r_0 - g + 1.$$

To get $\gamma(L) > \gamma_1$ we therefore require

$$(6.3) \quad 2(k-1)r_0 - g + 1 > k-2.$$

To prove this, note that $r_0 \geq \frac{\frac{g-5}{2}-k+3}{k-2}$ which is equivalent to

$$2(k-1)r_0 \geq g + \frac{1}{k-2}[g - (k-1)(2k-1)].$$

It is therefore sufficient to have

$$\frac{1}{k-2}[g - (k-1)(2k-1)] > k-3$$

which is true by our hypothesis. Proposition 6.1 implies the uniqueness of the line bundle computing γ_1 .

It remains to show that case (2) of Theorem 6.2 does not occur. By Lemma 2.8 this means that we must show that the map

$$H^0(K \otimes Q^*) \otimes H^0(K \otimes Q^*) \rightarrow H^0(K^2 \otimes Q^{*2})$$

is surjective. The argument of Remark 5.2 shows that $K \otimes Q^*$ is very ample. Now by [4, Theorem 2(a)] the map is surjective for

$$(6.4) \quad g > \max \left\{ \frac{k(k+1)}{2}, 10k-4 \right\}.$$

This inequality holds under our hypothesis on g . □

Remark 6.4. The number 46 as a lower bound for g is required only to ensure that (6.4) holds when $k = 5$.

Remark 6.5. Suppose $d_2 = 2d_1 - 1$ and $d_3 > 2d_1$. Then d_1 computes γ_1 (see (6.2)), $\gamma_2 = \gamma'_2 = \gamma_1$ and all bundles computing γ_2 also compute γ'_2 . Moreover, the proof of Proposition 6.1 shows that there is a unique line bundle Q of degree d_1 computing γ_1 and that $h^0(Q^2) = 3$. If Q is the only line bundle of degree $\leq g-1$ computing γ_1 , then the proof of Theorem 6.2 works with the same conclusion.

Remark 6.6. If $d_2 = 2d_1 - 2$, then d_1 computes $\gamma_1 = \frac{d_2}{2} - 1$. So again $\gamma_2 = \gamma'_2 = \gamma_1$. In this case the bundles E_L with L a line bundle of degree d_2 with $h^0(L) = 3$ compute γ_2 , but not γ'_2 .

Proposition 6.7. *If d_1 computes γ_1 and $d_2 \leq 2d_1 - 2$, the bundles computing γ_2 but not γ'_2 are precisely the bundles E_L , where L is a line bundle of degree d_2 with $h^0(L) = 3$, and all such bundles are stable.*

If d_1 computes γ_1 , $d_2 < 2d_1 - 2$ and $d_4 > d_2 + 2$, then these are the only bundles computing $\gamma_2 < \gamma'_2$.

Proof. Since $\gamma_1 = d_1 - 2$, the first statement follows from Corollary 2.4 and Lemma 2.2.

Now suppose $d_2 < 2d_1 - 2$ and $d_4 > d_2 + 2$; then $\frac{d_4}{2} - 2 > \frac{d_2}{2} - 1$. So $\gamma'_2 > \gamma_2$ by (2.1). Hence the bundles E_L are the only bundles computing γ_2 . \square

Remark 6.8. If $d_2 < 2d_1 - 2$ and $d_4 = d_2 + 2$, then the bundles E_L for L of degree d_2 still compute γ_2 . However there may exist further bundles computing simultaneously γ_2 and γ'_2 . If such bundles exist, then $\gamma'_2 < \gamma_1$. So they would give counterexamples to Mercat's conjecture.

7. GENERAL CURVES

For $g \leq 6$, Remark 5.3 and Propositions 5.7 and 5.10 apply to general curves. In this section we consider general curves of genus $g \geq 7$.

Proposition 7.1. *For a general curve of genus $g \geq 7$, $g \neq 8$, the bundles E_L with $d_L = d_2$ compute γ_2 and are the only bundles computing γ_2 .*

Proof. Recall that for a general curve,

$$(7.1) \quad \gamma_1 = \left\lfloor \frac{g-1}{2} \right\rfloor \quad \text{and} \quad d_r = r + g - \left\lfloor \frac{g}{r+1} \right\rfloor$$

(see [8, Remark 4.4(c)] and (2.5)). A simple numerical computation using (7.1) shows that $d_2 < 2d_1 - 2$ and $d_4 > d_2 + 2$. So Proposition 6.7 applies. \square

The general curve of genus 8 requires separate treatment because $d_2 = 2d_1 - 2$.

Proposition 7.2. *For a general curve of genus $g = 8$ the bundles computing $\gamma_2 = \gamma'_2 = 3$ are*

- (1) E_L with $d_L = d_2 = 8$;
- (2) $Q \oplus Q'$ with $d_Q = d_{Q'} = d_1 = 5$ and $h^0(Q) = h^0(Q') = 2$;
- (3) stable bundles E of degree 14 with $h^0(E) = 6$ given by a non-trivial extension

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q'^* \rightarrow 0$$

(such bundles exist when $Q \simeq Q'$).

Proof. The values of d_r come from (7.1). It follows from (2.1) that $\gamma_2 = \gamma'_2 = \gamma_1$. The bundles E_L and $Q \oplus Q'$ certainly have $\gamma(E) = \gamma_2$.

We have also $h^0(Q^2) = 3$ by Lemma 2.10 and it follows from Lemma 2.9 that there are no non-trivial extensions E of Q by Q with $h^0(E) = 4$.

If $Q \not\cong Q'$, then $h^0(Q \otimes Q') = 4$ since $d_4 = 11$. Again by Lemma 2.9 there are no non-trivial extensions E of Q by Q' with $h^0(E) = 4$.

It remains to determine whether there exist any semistable rank-2 bundles E of degree d_E with $2d_1 < d_E \leq 2g - 2$ and $\gamma(E) = \gamma_2$.

There are now 2 possibilities $d_E = 12$, $h^0(E) = 5$ and $d_E = 14$, $h^0(E) = 6$. Lemmas 2.5 and 2.6 imply that E must occur in an extension $0 \rightarrow Q \rightarrow E \rightarrow N \rightarrow 0$ with $\gamma(N) = \gamma_1$ and $h^0(N) = h^0(E) - 2$. If $d_E = 12$, we have $h^0(N) = 3$ giving $d_N \geq d_2 = 8$, a contradiction. If $d_E = 14$, then $d_N = 9$ and $h^0(N) = 4$. So N is the Serre dual of a line bundle Q' .

For the last statement we have to show according to Lemma 2.8 that the map

$$H^0(K \otimes Q^*) \otimes H^0(K \otimes Q^*) \rightarrow H^0(K^2 \otimes Q^{*2})$$

is not surjective. In this case the map factors through $S^2 H^0(K \otimes Q^*)$. However we have $\dim S^2 H^0(K \otimes Q^*) = 10$ and $h^0(K^2 \otimes Q^{*2}) = 11$.

It remains to prove stability of E . If E is not stable, it would have a line subbundle of degree 7 or 8. Since $d_2 = 8$, this implies that $h^0(E) \leq 5$, a contradiction. \square

We now consider the problem of finding bundles computing γ'_2 for general curves of genus $g \geq 7$, $g \neq 8$. It follows from (2.1) and (7.1) that $\gamma'_2 = \gamma_1$ if C is a general curve of genus ≤ 10 . This has also been proved for $g \leq 16$ in [3, Theorem 1.7] (it is a consequence of (2.1) and (7.1) for $g \leq 10$, $g = 12$ and $g = 14$). It is conjectured in [3] that this holds for general curves of arbitrary genus.

Lemma 7.3. *Suppose C is a general curve of genus $g \geq 7$. Then the only line bundles computing γ_1 of degree $\leq g - 1$ have degree d_1 , except when $g = 9$, where there are also bundles of degree d_2 computing γ_1 .*

Proof. For $g = 9$, we have $d_1 = 6$ and $d_2 = 8 = g - 1$. The result follows.

For $g \neq 9$, we have to show that $d_r - 2r > \gamma_1$ whenever $d_r \leq g - 1$ and $r > 1$.

By (7.1), the condition $d_r \leq g - 1$ is equivalent to $g \geq (r + 1)^2$. So we require to prove that this implies that

$$g - r - \left\lfloor \frac{g}{r+1} \right\rfloor > \left\lfloor \frac{g-1}{2} \right\rfloor.$$

It is sufficient to prove

$$g - r - \frac{g}{r+1} > \frac{g-1}{2},$$

which is equivalent to

$$g(r-1) > (2r-1)(r+1).$$

Since $g \geq (r+1)^2$, this is true for $r > 2$. For $r = 2$ we have $d_2 = g + 2 - \left\lfloor \frac{g}{3} \right\rfloor$. For $g = 7$ and 8 this is $> g - 1$. For $g \geq 10$, $d_2 > d_1 + 2$. So bundles of degree d_2 cannot compute γ_1 . \square

Theorem 7.4. *Let C be a general curve of genus $g \geq 7$, $g \neq 8$. Suppose that $\gamma'_2 = \gamma_1$. Then $\gamma_2 < \gamma'_2$ and the bundles computing γ'_2 are*

- (1) $Q \oplus Q'$ where Q, Q' are bundles computing γ_1 of degree d_1 or, if $g = 9$, of degree d_2 ;
- (2) possibly non-trivial extensions

$$0 \rightarrow Q \rightarrow E \rightarrow K \otimes Q'^* \rightarrow 0$$

where all sections of $K \otimes Q'^*$ lift to E ;

- (3) for g odd, non-trivial extensions

$$0 \rightarrow Q \rightarrow E \rightarrow Q' \rightarrow 0$$

where all sections of Q' lift to E (such extensions always exist when $Q \simeq Q'$ and $d_Q = d_1$);

- (4) possibly stable bundles not possessing a line subbundle with $h^0 \geq 2$. We have necessarily $h^0(E) = 2 + s$ with

$$(7.2) \quad 2 \leq s \leq \begin{cases} \frac{g-1}{2} & \text{if } g \text{ is odd,} \\ \frac{g-2}{4} & \text{if } g \text{ is even} \end{cases}$$

$$\text{and } d_E = 2\gamma_1 + 2s.$$

Proof. We have $\gamma_2 < \gamma'_2$ by (2.1) and (7.1). Clearly the bundles of type (1) compute $\gamma'_2 = \gamma_1$.

If E computes γ'_2 and E has a subbundle M with $h^0(M) \geq 2$, then by Lemma 2.6 we must have an extension $0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$ with $\gamma(M) = \gamma(N) = \gamma_1$ and all sections of N lift to E . In view of Lemma 7.3 the only possibilities are types (2) and (3).

For $g \geq 10$ even, it follows from Lemmas 2.9 and 2.10 that there exists no non-trivial extension of type (3) with $Q \simeq Q'$. If $Q \not\simeq Q'$, it follows from [16, Proposition 4.1] that $h^0(Q \otimes Q') = 4$ and Lemma 2.9 applies again. For g odd, the existence of non-trivial extensions when $Q \simeq Q'$ follows again from Lemmas 2.9 and 2.10.

If E computes γ'_2 and E does not admit a subbundle with $h^0 \geq 2$, then $h^0(E) = 2 + s$ with $s \geq 2$, $d_E = 2\gamma_1 + 2s \leq 2g - 2$ and $d_E \geq d_{2s}$ by Lemma 2.5. Any quotient line bundle L of E must have $h^0(L) \geq s + 1$. So $d_L \geq d_s$. If E is strictly semistable, then $d_E \geq 2d_s$, giving $\gamma_1 + s \geq d_s$. Since $\gamma_1 = d_1 - 2$, this contradicts the fact that $d_s \geq d_1 + s - 1$. Hence E is stable.

It remains to prove (7.2). By (7.1) we have

$$2 \left\lfloor \frac{g-1}{2} \right\rfloor + 2s \geq g + 2s - \left\lfloor \frac{g}{2s+1} \right\rfloor,$$

which is equivalent to the second inequality of (7.2). \square

Remark 7.5. Semistable bundles of type (2) exist when $Q \simeq Q'$ and $g = 7, 9$ or 11 by dimensional calculations using Lemma 2.8. Such bundles do not exist when $Q \simeq Q'$ and $g \geq 10$, $g \neq 11$, by [16, Propositions 4.2 and 4.3].

Remark 7.6. For all odd g , there exist bundles of type (4) with $s = 2$ [3, Theorem 1.1] (see also [5, postscript]). Provided that $\gamma'_2 = \gamma_1$, these bundles compute γ'_2 .

Proposition 7.7. *Let C be a general curve of genus 7. Bundles of type (4) exist for $s = 3$.*

Proof. If $s = 3$, we have $\gamma_1 = 3$ and $d_E = 12$. By Lemma 2.5, $h^0(\det E) \geq 7$. So $\det E \simeq K$. By [13] the semistable E possesses a subbundle M of degree ≥ 3 . Considering cases, we see that we must have an extension

$$(7.3) \quad 0 \rightarrow M \rightarrow E \rightarrow K \otimes M^* \rightarrow 0$$

with $h^0(M) = 1$, $d_M = 3$, $h^0(K \otimes M^*) = 4$ and all sections of $K \otimes M^*$ lift to E . By Lemma 2.8 and the fact that $h^0(K^2 \otimes M^{*2}) = 12$ by Riemann-Roch, we see that there exists such an extension for every M .

Suppose that E is not semistable. By considering cases, we see that there exists an extension

$$0 \rightarrow M' \rightarrow E \rightarrow K \otimes M'^* \rightarrow 0$$

with $d_{M'} = 7$ and $h^0(M') = 3$. Hence there exists a nonzero homomorphism $M \rightarrow K \otimes M'^*$. Since $K \otimes M'^*$ is a generated line bundle of degree 5 with $h^0(K \otimes M'^*) = 2$, there is at most a 1-dimensional system of such line bundles M for any fixed M' . Moreover, M' belongs to the Brill-Noether locus of line bundles of degree 7 with $h^0 \geq 3$ which on a general curve of genus 7 has dimension 1. So the system of M for which such an M' exists has dimension at most 2. Hence for a general M for which an extension (7.3) exists there is no such M' . \square

Proposition 7.8. *Let C be a general curve of genus 9. Then bundles of type (4) with $s = 4$ exist.*

Proof. The proof is similar to the proof of Proposition 7.7. In this case, $d_M = 4$, $h^0(K \otimes M^*) = 5$ and $h^0(K^2 \otimes M^{*2}) = 16$. So Lemma 2.8 applies again. \square

Remark 7.9. If $g = 9$ and $s = 3$, we are not able to decide whether any bundles of type (4) exist. If g is odd ≥ 11 , the argument of Propositions 7.7 and 7.8 no longer works, even in the case $s = \frac{g-1}{2}$.

Proposition 7.10. *Let C be a general curve of genus 10. Then $\gamma_2 < \gamma'_2 = \gamma_1$ and the only bundles computing γ'_2 are the bundles $Q \oplus Q'$, where Q and Q' are bundles of degree 6 with $h^0 = 2$.*

Proof. In this case, if E is a bundle of type (4), we must have $s = 2$ and $d_E = 12$. There do not exist any stable bundles E of this type by [5, Theorem 4.1(i)]. For bundles of type (2), it follows from [5, postscript] that

$$H^0(K \otimes Q'^*) \otimes H^0(K \otimes Q^*) \rightarrow H^0(K^2 \otimes Q^* \otimes Q'^*)$$

is always surjective. Hence, by Lemma 2.8, there are no extensions of type (2). This leaves only type (1). \square

8. CURVES WITH $\gamma'_2 < \gamma_1$

We have already noted that a general curve of genus $g \leq 16$ has $\gamma'_2 = \gamma_1$ and it is conjectured that this holds for general curves of arbitrary genus. However there are examples of curves of any genus $g \geq 11$ for which $\gamma'_2 < \gamma_1$ (see [3]).

In this section we shall refer to stable bundles E with $h^0(E) = 2 + s$, $s \geq 2$, not possessing a line subbundle with $h^0 \geq 2$ as *bundles of type PR*.

We begin by considering bundles computing γ_2 .

Proposition 8.1. *Suppose $\gamma'_2 < \gamma_1$. Then $\gamma_2 = \frac{d_2}{2} - 1$ and the bundles computing γ_2 but not γ'_2 are precisely the bundles E_L where L is a line bundle with $d_L = d_2$ and $h^0(L) = 3$.*

Proof. It follows from (2.1) that $\gamma_2 = \frac{d_2}{2} - 1 < \gamma_1$. The result now follows from Corollary 2.4. \square

We turn now to the consideration of bundles computing γ'_2 .

Proposition 8.2. *Suppose $\gamma'_2 < \gamma_1$. Then all bundles computing γ'_2 are of type PR with*

$$2 \leq s \leq \gamma'_2 - \frac{\gamma_1}{2}.$$

Proof. If E is a semistable bundle computing γ'_2 and possessing a line bundle M with $h^0(M) \geq 2$, then by Lemma 2.6, $\gamma(E) \geq \gamma_1$, a contradiction. So E is of type PR and

$$d_{2s} \leq d_E \leq 2g - 2.$$

Since $d_{g-1} = 2g - 2$, this implies that $2s \leq g - 1$. By (2.2) we have

$$d_{2s} \geq \min\{\gamma_1 + 4s, g + 2s - 1\}.$$

If $g - 1 - \gamma_1 \leq 2s \leq g - 1$, this gives $d_{2s} \geq g + 2s - 1$, but

$$d_{2s} \leq g + 2s - \left\lfloor \frac{g}{2s+1} \right\rfloor = g + 2s - 1.$$

So $d_{2s} = g + 2s - 1$ and $\gamma'_2 = \gamma(E) \geq \frac{g-1}{2}$, a contradiction.

Hence $2s < g - 1 - \gamma_1$ and $d_{2s} \geq \gamma_1 + 4s$ which implies

$$\gamma'_2 = \gamma(E) \geq \frac{\gamma_1 + 2s}{2} = \frac{\gamma_1}{2} + s.$$

So $s \leq \gamma'_2 - \frac{\gamma_1}{2}$. Stability of E follows as in the proof of Theorem 7.4, since $d_1 - 2 \geq \gamma_1$. \square

Theorem 8.3. *Suppose $\gamma'_2 < \gamma_1$ and $d_4 = 2\gamma'_2 + 4$. Then the set of bundles of type PR with $s = 2$ which compute γ'_2 is in bijective correspondence with the set of line bundles*

$$U(d_4, 5) := \left\{ M \mid \begin{array}{l} d_M = d_4, \quad h^0(M) = 5, \\ S^2 H^0(M) \rightarrow H^0(M^2) \text{ not injective} \end{array} \right\}.$$

Proof. If E is a bundle of type PR with $s = 2$ which computes γ'_2 , then

$$d_E = 2\gamma'_2 + 4 = d_4 < 2d_1.$$

Since E is necessarily stable, the result follows from [5, Theorem 3.2 and Remark 3.4]. \square

The following corollary generalises [7, Proposition 4.5]

Corollary 8.4. *Suppose $\gamma_1 \geq 5$ and $\gamma'_2 = \frac{\gamma_1}{2} + 2$. Then $U(d_4, 5)$ is non-empty and the corresponding bundles of type PR are the only bundles that compute γ'_2 .*

Proof. By Proposition 8.2, every bundle E computing γ'_2 is of type PR with $s = 2$. By (2.1), $\gamma'_2 \geq \frac{d_4}{2} - 2$. So $d_4 \leq 2\gamma'_2 + 4$.

On the other hand, $d_4 \geq \gamma_1 + 8$ by (2.2). So $d_4 \geq 2\gamma'_2 + 4$ and hence

$$d_4 = 2\gamma'_2 + 4.$$

The result follows from the theorem. \square

Corollary 8.5. *Suppose $\gamma'_2 < \gamma_1 = 5$. Then $U(d_4, 5)$ is non-empty and the corresponding bundles of type PR are the only bundles that compute γ'_2 .*

Proof. In this case $\gamma'_2 = \frac{9}{2} = \frac{\gamma_1}{2} + 2$, since $\gamma'_2 < \gamma_1 = 5$ and $\gamma'_2 \geq \frac{9}{2}$ by (2.1). So the assertion follows from the previous corollary. \square

Proposition 8.6. *Suppose $\gamma_1 \geq 6$ and $\gamma'_2 = \frac{\gamma_1 + 5}{2}$. Then the bundles computing γ'_2 are all of type PR with $s = 2$ and $d_E = 2\gamma'_2 + 4$. The set of such bundles is in bijective correspondence with*

$$U^0(2\gamma'_2 + 4, 5) := \{M \in U(2\gamma'_2 + 4, 5) \mid M \text{ generated}\}.$$

Proof. We have $s = 2$ by Proposition 8.2. By (2.2) we have $d_5 \geq \gamma_1 + 10 > 2\gamma'_2 + 4$. Then the argument of [5, Theorem 3.2 and Remark 3.4] gives the assertion. \square

Corollary 8.7. *Suppose $\gamma'_2 < \gamma_1 = 6$. Then $\gamma'_2 = 5$ or $\frac{11}{2}$ and $d_4 = 14$ or 15.*

- (1) *If $\gamma'_2 = 5$, then the set of bundles computing γ'_2 is in bijective correspondence with $U(14, 5)$.*
- (2) *If $\gamma'_2 = \frac{11}{2}$, then the set of bundles computing γ'_2 is in bijective correspondence with $U^0(15, 5)$.*

If $d_4 = 15$, then $\gamma'_2 = \frac{11}{2}$ and $U^0(15, 5) = U(15, 5)$.

Proof. Since $\gamma'_2 \geq \frac{\gamma_1}{2} + 2$, we must have either $\gamma'_2 = 5$ or $\gamma'_2 = \frac{11}{2}$. Since $\gamma_1 + 8 \leq d_4 \leq 2\gamma'_2 + 4$, we have $d_4 = 14$ or 15 .

If $\gamma'_2 = 5$, the result follows from Theorem 8.3. If $\gamma'_2 = \frac{11}{2}$, it follows from Proposition 8.6. \square

Example 8.8. We know from [3] that there exist examples of curves of genus 11 and 12 with $\gamma_1 = 5$ and $\gamma'_2 = \frac{9}{2}$. So Corollary 8.5 applies.

Example 8.9. From the same source we know that there exist curves of genus 13 with $\gamma_1 = 6$ and $\gamma'_2 \leq \frac{11}{2}$. So Corollary 8.7 applies. We are not certain whether there exist curves of genus 13 with $\gamma_1 = 6$ and $\gamma'_2 = 5$.

Example 8.10. By [10, Theorem 1.1] there exists a curve of genus 14 with $\gamma_1 = 6$ and $\gamma'_2 = 5$ provided that the quadratic form

$$3m^2 + 14mn + 13n^2$$

cannot take the value -1 for any integers m and n .

If the quadratic form does take the value -1 , then reduction modulo 4 shows that m is odd and n is even. Writing $m = 2m_1 + 1$ and $n = 2n_1$ reduces the equation to

$$3(m_1^2 + m_1) + 7n_1(2m_1 + 1) + 13n_1^2 = -1.$$

The left hand side of this equation is always even, a contradiction. So the curve exists.

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